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Dyson's Coulomb gas on a circle and intermediate eigenvalue statistics

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Abstract. Dyson's two-dimensional Coulomb gas on the unit circle with inverse temperature β is investigated with the help of a Metropolis algorithm. Theoretical predictions for energy and specific heat are verified. The connection with the theory of ensembles of random unitary matrices with orthogonal, unitary or symplectic symmetry corresponding to $\beta = 1, 2, 4$ is investigated in detail. New approximation formulae for the spacing distribution and the so-called delta statistics are proposed and found to be useful, not only in the three cases mentioned but also for all β between 0 and 4. Our approach does not give a fitting parameter like the Brody distribution but an approximation to the true β. Moreover, it is especially useful in cases when the application of the Berry–Robnik distribution is baseless, namely for quantum systems with a completely chaotic classical limit that show intermediate eigenvalue statistics because of Anderson localisation.

1. Introduction

Currently an overwhelmingly large number of examples exist which prove the usefulness of random matrix theory (RMT) applied to quantum systems which are chaotic in the classical limit (Eckhardt 1988). RMT establishes the existence of three spectral universality classes connected with the three groups $O(N)$, $U(N)$ and, for even N, $Sp(N)$, which leave ensembles of unitary or Hermitian $N \times N$ matrices invariant. The three ensembles of unitary matrices, which are already uniquely defined by this invariance property, were introduced by Dyson (1962a), who called them circular ensembles. They are commonly abbreviated COE, CUE and CSE. The corresponding ensembles of Hermitian $N \times N$ matrices need an additional specification to be unique (namely independent Gaussian distributions of the matrix elements) and are usually abbreviated GOE, GUE and GSE. The most prominent difference between these ensembles is the degree β of eigenvalue repulsion, which depends only on the underlying symmetry group and is linear ($\beta = 1$) for COE and GOE, quadratic ($\beta = 2$) for CUE and GUE, and finally quartic ($\beta = 4$) for CSE and GSE.

While the Gaussian ensembles turned out to be appropriate for describing the spectral properties of 'typical' Hamiltonians with discrete spectra, which do not possess enough 'good' quantum numbers to diagonalise them in a physically simple basis,

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the same holds true for circular ensembles and 'typical' unitary propagators of non-autonomous quantum dynamics. A fourth possibility of spectral behaviour, Poissonian distribution of eigenvalues, showing no repulsion ($\beta = 0$), seemed to be connected with quantum systems with an integrable classical limit and more than one degree of freedom (Berry and Tabor 1977).

As the universal features of quantum systems concerning their eigenvalues (spacing distribution, spectral stiffness, fluctuation in the spectral staircase) and distribution of eigenvector components, and their relations to the symmetries of the quantum systems in question are now well understood, the 'non-universal' properties are beginning to receive attention. Such properties include, for example, the saturation of the stiffness and fluctuations of the spectral staircase and other peculiarities of quantum spectra depending on the short periodic orbits of the corresponding classical system, which are of course non-universal (Berry 1985).

In this paper we turn instead to another class of experimental results that are in contrast to the mentioned picture of three universality classes, namely intermediate eigenvalue statistics, which can be described by a repulsion parameter β ranging between 0 and 4. Two groups of quantum systems show that there is need for intermediate values of β. For quantum systems that are not completely chaotic in the classical limit one typically finds spacing distributions and so-called Δ statistics, measuring the spectral stiffness, that lie between two extremes, the Poissonian case ($\beta = 0$) and the maximal quantum chaos (Izrailev 1986) which is attained in the case of the completely chaotic classical limit with $\beta = 1$, 2 or 4. On the other hand, a completely chaotic classical limit does not guarantee this limiting quantum behaviour, as is known from the kicked rotator. Localisation of the perturbed eigenfunctions in the unperturbed representation is the reason for suppression of maximal quantum chaos accompanied by intermediate β (Izrailev 1984, 1986, Feingold and Fishman 1987, Frahm and Mikeska 1988, Izrailev 1988).

Two main ways are known to describe intermediate spacing distributions: the Brody distribution (Brody *et al* 1981) and the Berry–Robnik distribution (Berry and Robnik 1984). The former is nothing but an interpolation formula to fit numerical results and only a limited amount of information can be drawn from the value of the fit parameter. Moreover, the Brody distribution is not appropriate to fit spectra which are intermediate in their properties between the orthogonal ($\beta = 1$) and the symplectic ($\beta = 4$) case. The Berry–Robnik distribution, on the other hand, was deduced under the assumption of co-existing regular and chaotic regions in classical phase space. This is not the case for the strongly kicked rotator which nevertheless can show intermediate eigenvalue statistics.

We propose to go back to Dyson's original idea of using a connection between the eigenvalue distribution for the three circular ensembles with $\beta = 1, 2, 4$ and the partition function of a two-dimensional Coulomb gas on the unit circle with inverse temperature β. Dyson (1962b) used this connection to find the asymptotic behaviour of the spacing distributions for large spacings between neighbouring particles in the three cases mentioned. But now we have an application for the intermediate cases with $0 \le \beta \le 2$, namely the localised kicked rotator with broken time-reversal symmetry (Izrailev 1986, 1987, 1988). For a generalisation to spin-dependent kicking potentials (Scharf 1989) one finds $0 \le \beta \le 4$.

First we collect some results of random matrix theory (RMT) concerning eigenvalues. Then we briefly review Dyson's work on the Coulomb gas on the circle and verify some theoretical results with the help of a Monte Carlo simulation. We state conjectures on intermediate spacing distributions and Δ statistics and investigate them numerically in detail for $\beta = 1, 2, 4$ and over the whole range between 0 and 4. We use the conjectures to propose a new method to find the degree of level repulsion for experimental spectra, and discuss its usefulness for the investigation of quantum systems with (partially) localised eigenstates and a strongly chaotic classical limit.

2. Spectral properties of unitary random matrices

In this section we collect some results of RMT concerning eigenphases of unitary $N \times N$ matrices drawn from the ensembles COE or CUE or of $2N \times 2N$ matrices drawn from the symplectic ensemble CSE. The eigenphases will be denoted by θ_k with $k = 1, 2, \ldots, N$. In the symplectic case eigenvalues are twofold degenerate, so it suffices to give N eigenphases in this case, too.

The joint distribution of eigenphases for the circular ensembles COE, CUE and CSE ($\beta = 1, 2, 4$, respectively) is exactly given by (Mehta 1967)

$$Q_{N\beta}(\theta_1, \ldots, \theta_N) = Z_{N\beta}^{-1} \prod_{1 \le j < k \le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{\beta} \tag{2.1}$$

with the normalisation constant $Z_{N\beta}$

$$Z_{N\beta} = (2\pi)^{-N} \int_0^{2\pi} \prod_{1 \le j < k \le N} \left| e^{i\theta_j} - e^{i\theta_k} \right|^{\beta} d\theta_1 \ldots d\theta_N \tag{2.2}$$

given by Dyson (1962a)

$$Z_{N\beta} = \Gamma(1 + \beta N/2) \left[\Gamma(1 + \beta/2) \right]^{-N}. \tag{2.3}$$

The eigenphases θ_j possess a mean next-nearest-neighbour (NNN) spacing of $2\pi/N$, but the distribution of normalised NNN spacings S, called $P(S)$, is known in closed form only for a few special cases. For $N \to \infty$ expansions in powers of S are possible (Mehta 1967, Dietz 1989); for example, up to order S^7 (Dietz and Haake 1990):

$$\begin{aligned} P_1(S) &= \tfrac{1}{6}\pi^2 S - \tfrac{1}{60}\pi^4 S^3 + \tfrac{1}{270}\pi^4 S^4 + \tfrac{1}{1680}\pi^6 S^5 - 0.2035 S^6 - 0.1046 S^7 \\ P_2(S) &= \tfrac{1}{3}\pi^2 S^2 - \tfrac{2}{45}\pi^4 S^4 + \tfrac{1}{315}\pi^6 S^6 - 0.2347 S^7 \\ P_4(S) &= 11.5448 S^4 - 26.0440 S^6. \end{aligned} \tag{2.4}$$

For $N \to \infty$ Dyson (1962b) deduced the asymptotic behaviour of $P_\beta(S)$ for large S

$$P_\beta(S) \sim A\, S^{2+(1+\beta/2)^2/2\beta} \exp[-\tfrac{1}{16}\pi^2 \beta S^2 - \tfrac{1}{2}\pi(1 - \beta/2)S]. \tag{2.5}$$

To have an approximation for $P_\beta(S)$ in a simple form it has become customary to use the Wigner surmise

$$P_\beta(S) \simeq C_\beta S^\beta \exp(-A_\beta S^2) \tag{2.6}$$

with the constants A_β and C_β to be determined by normalisation and setting the mean of S to 1: $\langle 1\rangle = \langle S\rangle = 1$. For the physically interesting cases one finds

$$
\begin{aligned}
P_1(S) &\simeq \frac{\pi}{2} S \exp\left(-\frac{\pi}{4} S^2\right) \\
P_2(S) &\simeq \frac{32}{\pi^2} S^2 \exp\left(-\frac{4}{\pi} S^2\right) \\
P_4(S) &\simeq \left(\frac{64}{9\pi}\right)^3 S^4 \exp\left(-\frac{64}{9\pi} S^2\right).
\end{aligned}
\tag{2.7}
$$

Comparing the approximation (2.7) with the expansions (2.4) shows that even the slopes at $S = 0$ come out wrong. Nevertheless, the total error one makes by using (2.7) is less than 5%, which is enough for most practical purposes because it can be detected only with a sample containing more than about 10 000 spacings.

Another spectral property in respect of which the three ensembles behave differently is the so-called number variance which measures the fluctuations in the number $n(L)$ of eigenvalues that a strip of length L contains:

$$
\Sigma_\beta^2(L) = \langle (n(L) - L)^2 \rangle \qquad \langle n(L) \rangle = L. \tag{2.8}
$$

Pandey (1979) gives, up to $O(1/L)$,

$$
\begin{aligned}
\Sigma_1^2(L) &= \frac{2}{\pi^2}\left(\ln(2\pi L) + \gamma + 1 - \frac{\pi^2}{8}\right) \\
\Sigma_2^2(L) &= \frac{1}{\pi^2}[\ln(2\pi L) + \gamma + 1] \\
\Sigma_4^2(L) &= \frac{1}{2\pi^2}\left(\ln(4\pi L) + \gamma + 1 + \frac{\pi^2}{8}\right)
\end{aligned}
\tag{2.9}
$$

with γ denoting Euler's constant: $\gamma = 0.577\,215\,66\ldots$.

Finally we give expressions for the so-called Δ statistics (or sometimes Δ_3 statistics, for historical reasons) which measures the stiffness of the spectral staircase

$$
\Delta_\beta(L) = \frac{1}{L} \max \int_{\theta_0}^{\theta_0 + 2\pi L/N} [N(\theta) - A\theta - B]^2 \, d\theta \tag{2.10}
$$

which should be minimised by changing A and B. It can be shown (Pandey 1979) that there exists a connection between $\Delta_\beta(L)$ and $\Sigma_\beta^2(L)$ up to $O(1/L)$

$$
\Delta_\beta(L) = \frac{1}{2}\Sigma_\beta^2(L) - \frac{9}{4\beta\pi^2} \tag{2.11}
$$

and from this up to $O(1/L)$

$$
\begin{aligned}
\Delta_1(L) &= \frac{1}{\pi^2}\left(\ln(2\pi L) + \gamma - \frac{\pi^2}{8} - \frac{5}{4}\right) \\
\Delta_2(L) &= \frac{1}{2\pi^2}\left(\ln(2\pi L) + \gamma - \frac{5}{4}\right) \\
\Delta_4(L) &= \frac{1}{4\pi^2}\left(\ln(4\pi L) + \gamma + \frac{\pi^2}{8} - \frac{5}{4}\right).
\end{aligned}
\tag{2.12}
$$

For reasons of completeness, we note the corresponding expressions for the Poissonian spectrum, formally setting $\beta = 0$:

$$P_0(S) = \mathrm{e}^{-S} \qquad \Sigma_0^2(L) = L \qquad \Delta_0(L) = L/15. \tag{2.13}$$

All these results have been verified for spectra of autonomous (Haller *et al* 1984, Seligman *et al* 1984, Berry and Robnik 1986) as well as kicked quantum dynamics (Izrailev 1986, Frahm and Mikeska 1986, Nakamura *et al* 1986, Haake *et al* 1987, Kuś *et al* 1987) with a strongly chaotical classical limit for all cases $\beta = 1, 2$ and 4, although in the latter case up to now only for kicked dynamics (Scharf *et al* 1988, Scharf 1989).

Some of the previous formulae contain β explicitly and the question arises as to whether they are valid not only for $\beta = 1$, 2 and 4 but also for intermediate β. This is the case for the normalisation constant $Z_{N\beta}$ (2.3). That the formula (2.5), which describes the behaviour of $P_\beta(S)$ for $S \gg 1$, also makes sense for intermediate β was shown by Dyson with the help of a thermodynamical model, which is introduced in the next section.

3. Dyson's Coulomb gas on the unit circle

Dyson (1962b) introduced the model of a gas of N equally charged particles moving on the unit circle and interacting via a Coulomb force in two dimensions. His aim was to investigate the properties of the phase distribution $Q_{N\beta}(\theta_1, \ldots, \theta_N)$ mentioned in section 2. The connection between the Coulomb gas and the distribution of eigenphases of the circular ensembles becomes obvious when looking at the partition function of the Coulomb gas

$$Z_N(\beta) = (2\pi)^{-N} \int_0^{2\pi} \mathrm{e}^{-\beta W}\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N \tag{3.1}$$

with the potential energy W

$$W = - \sum_{1 \le j < k \le N} \ln |\mathrm{e}^{\mathrm{i}\theta_j} - \mathrm{e}^{\mathrm{i}\theta_k}|. \tag{3.2}$$

The trivial momentum-dependent part of the partition function has been discarded. For $\beta = 1$, 2 and 4, $Z_N^{-1}(\beta)$ is the normalisation constant of the joint distribution of eigenphases $Q_{N\beta}(\theta_1, \ldots, \theta_N)$ of the circular ensembles. But for the Coulomb gas β is allowed to take on any positive value. The minimum potential energy that the gas acquires in the zero-temperature limit (i.e. $\beta \to \infty$) can be shown to be $W_0 = -\frac{1}{2} N \ln N$. To have a finite expression for the potential energy in the thermodynamic limit ($N \to \infty$), one defines a new energy scale and a new partition function

$$\Phi_N(\beta) = (2\pi)^{-N} \int_0^{2\pi} \exp\left[-\beta(W - W_0)\right]\, \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N. \tag{3.3}$$

Using (2.3) for $Z_N(\beta)$ one finds for the free energy per particle in the thermodynamic limit (Dyson 1962b, Mehta 1967)

$$F(\beta) = \frac{L(\beta/2)}{\beta} + \frac{1 - \ln(\beta/2)}{2} \tag{3.4}$$

with the abbreviation $L(z) = \ln\Gamma(1+z)$. From this one calculates for the potential energy and specific heat per particle

$$U(\beta) = \tfrac{1}{2}[L'(\beta/2) - \ln(\beta/2)] \qquad C(\beta) = -\tfrac{1}{4}\beta^2 L''(\beta/2) + \beta/2. \tag{3.5}$$

These last two formulae we need later for comparison with Monte Carlo simulations. Dyson's main interest in investigating the Coulomb gas was to find an expression for the spacing distribution $P_{N\beta}(S)$ for $N \to \infty$. He only partially succeeded in finding the asymptotic behaviour for $S \gg 1$ (or more precisely $\beta S \gg 1$) which we have already given in (2.5).

Before closing this section, two general statements should be made. Allowing for further interaction terms between the particles on the circle, for example (Yukawa 1986)

$$W_1 = -\frac{1}{2} \sum_{1 \le j < k \le N} \ln\{\alpha_0 + \alpha_1 \sin^2[(\theta_j - \theta_k)/2]\}, \tag{3.6}$$

might change their distribution function, although a large class of interaction terms does not have any influence in the thermodynamic limit (Kamien *et al* 1988). It was already recognised by Dyson that the same Coulomb gas constrained to move on an infinite straight line with an additional confining harmonic potential has a partition function of the same form as the normalisation integral of the joint eigenvalue distribution function of the Gaussian ensembles.

In this paper we do not investigate the distribution of eigenvector components for the circular ensembles because the mapping to the Coulomb gas only involves the eigenvalues. But a generalisation of the results for the universal cases $\beta = 1$, 2 and 4 (Izrailev 1987, Kuś *et al* 1988) to the intermediate values is of great interest.

4. Intermediate spacing distribution and Δ statistics

Since, for $\beta = 0$, 1, 2 and 4, the spacing distribution $P_\beta(S)$ for a quantum dynamics with a chaotic classical limit behaves like S^β for $S \ll 1$, it is tempting to assume that this holds true for quantum systems showing intermediate spacing distribution with conveniently chosen positive β. But the non-analytic behaviour of S^β in $S = 0$ contradicts the results from almost degenerate perturbation theory, namely that the degree β of level repulsion depends only on the symmetries of the quantum dynamics in the subspace spanned by the near degenerate eigenfunctions which leads to $\beta = 1$, 2, and 4 or to $\beta = 0$ in the case of a perturbation with vanishing coupling between the eigenfunctions (Scharf *et al* 1988). Therefore $P_\beta(S) \sim S^\beta$ with a non-universal β can be true at best on an intermediate scale $0 < S_0 \le S \ll 1$. On a finer scale the universal behaviour with $\beta = 0$, 1, 2 or 4 will finally dominate. In experiments only the intermediate scale is resolvable and therefore $P_\beta(S) \sim S^\beta$ might be worth testing.

One of us has recently proposed (Izrailev 1988) using Dyson's asymptotic result (2.5) for large S in slightly modified form together with the behaviour discussed for small S to obtain the following approximation for $P_\beta(S)$ for the whole range of spacings $0 < S < \infty$:

$$P_\beta(S) = A\left(\frac{\pi S}{2}\right)^\beta \exp\left[-\frac{\beta\pi^2}{16}S^2 - \left(B - \frac{\beta\pi}{4}\right)S\right] \tag{4.1}$$

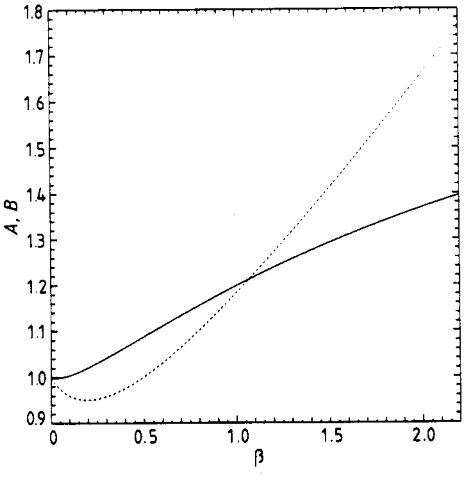

Figure 1. Dependence of the constants A (full curve) and B (broken curve) in spacing distribution (4.1) upon β.

with the β-dependent constants A and B to be determined from normalisation and from $\langle S \rangle = 1$. Figure 1 shows their dependence on β for the range of largest physical interest. The values of A and B for $\beta = 1, 2$ and 4 are: $A_1 = 1.198\ldots$, $B_1 = 1.183\ldots$, $A_2 = 1.369\ldots$, $B_2 = 1.658\ldots$, $A_4 = 1.551\ldots$, $B_4 = 2.711\ldots$. Several examples for the form of the distribution itself will be given in section 7.

Compared with the Brody distribution, the conjectured distribution (4.1) has the advantage that it shows the correct asymptotic behaviour in $S \to \infty$ for $\beta > 1$ and that β is not just a fitting parameter but has physical meaning, as will be shown in section 7. In contrast to the Berry–Robnik distribution, it provides the possibility of having an intermediate spacing distribution without relying on classical phase space structures. This is important because, for example, the strongly kicked rotator, although it does not possess any non-chaotic structures of mentionable weight in the classical phase space, nevertheless shows intermediate statistics ($0 < \beta < 1$ with time-reversal invariance or $0 < \beta < 2$ without) which is in contrast to the prediction of the Berry–Robnik distribution. The explanation is that the eigenfunctions show Anderson localisation, an effect that is not accounted for in Berry–Robnik distribution. In addition to that the latter does not, in general, vanish at $S = 0$, which causes strong deviations for small S between experimental histograms and the Berry–Robnik distribution. This can only be remedied by additional *ad hoc* requirements (Robnik 1987).

Turning now to the formulae (2.9) and (2.12) for $\Sigma^2_\beta(L)$ and $\Delta_\beta(L)$, the number variance and the Δ statistics, the following conjectures for $0 < \beta \leq 4$ are tempting (up to $\mathrm{O}(1/L)$):

$$\begin{aligned}\Sigma^2_\beta(L) &= \frac{2}{\beta\pi^2}\ln(L) + \Sigma^2_\beta(1)\\ \Delta^2_\beta(L) &= \frac{1}{\beta\pi^2}\ln(L) + \Delta^2_\beta(1).\end{aligned} \tag{4.2}$$

For $\beta \to 0$ these approximate formulae do not give the correct linear Poissonian behaviour. Therefore one expects that they fail when β is too small.

5. Monte Carlo simulation of Dyson's Coulomb gas on the circle

To check the stated conjectures concerning the spacing distribution, the number variance and the Δ statistics, we made a Monte Carlo simulation of Dyson's Coulomb gas on the unit circle with $N = 99$ particles and inverse temperature β. We describe the Metropolis algorithm used (see, for example, Binder and Heermann 1988) only briefly. We started with different initial distributions of the particles on the circle (equally spaced, Poissonian-distributed spacings, positions calculated from eigenvalues of a random unitary matrix) to check the independence of our results from the initial distribution. The evolution of the dynamics was made in discrete time steps, each step containing the following substeps.

(i) Calculate the energy $W = W(\theta_1, \ldots, \theta_N)$ given by (3.2) for the set of particle positions $(\theta_1, \ldots, \theta_N)$.

(ii) Draw N random numbers $(r_1, \ldots, r_N)$, equally distributed in the interval $[-r, r]$ with $r = 2\pi/N(\text{Noise})$ (we present results for Noise = 0.1 and 0.2).

(iii) Compute positions $\theta'_k = \theta_k + r_k$ $(k = 1, \ldots, N)$. Because of the logarithmic singularities in W, the particle positions cannot change their order. Therefore, only sets of positions $(\theta'_1, \ldots, \theta'_N)$ that respect the original order are allowed.

(iv) The energy $W' = W(\theta'_1, \ldots, \theta'_N)$ is calculated. If $W' < W$, the particle positions take on the new values $(\theta'_1, \ldots, \theta'_N)$. But if $W' > W$, then the Boltzmann factor $q = e^{\beta(W-W')} < 1$ will be compared with another random number ρ, now equally distributed in $[0, 1]$. If $q > \rho$ the particle positions will take on the new values $(\theta'_1, \ldots, \theta'_N)$, too. Otherwise the system remains unchanged.

After waiting an appropriately chosen number of time steps for the system to relax into equilibrium, the sets of positions, generated in the course of time, can be used to calculate quantities of physical interest. The sampling of the positions should be done at times far enough apart for correlations to decay sufficiently.

First we investigated the β dependence of the potential energy $U(\beta)$ and the specific heat per particle $C(\beta)$ and compared them with the theoretical predictions (3.5) to test the reliability of our simulation procedure. The results in figure 2 show that the experimental energies follow the theoretical prediction closely. Only for small β is there a systematic deviation which can be explained by the finite-size effects and prohibitively long relaxation times. As the specific heat $C(\beta)$ is a measure of the fluctuation of the energy $U(\beta)$ per particle in an infinite system, we expect for our small system only qualitative agreement between theory and experiment. With this restriction in mind the results shown in figure 2 are quite satisfactory. The behaviour of the specific heat and the potential energy upon changing β shows that our Monte Carlo simulations give reliable results. A clearcut dependence of these results on the degree of noise we used is not visible.

After comparing the numerical results with quantities known analytically, we now proceed to quantities whose dependence on the inverse temperature β is not known in closed form.

6. Results for $\beta = 1$, 2 and 4

We want to compare the results of our Monte Carlo simulation for $N = 99$ particles (or eigenphases) at an inverse temperature $\beta = 1$, 2 and 4 with the properties of the three corresponding universality classes from RMT. We focus our attention on the

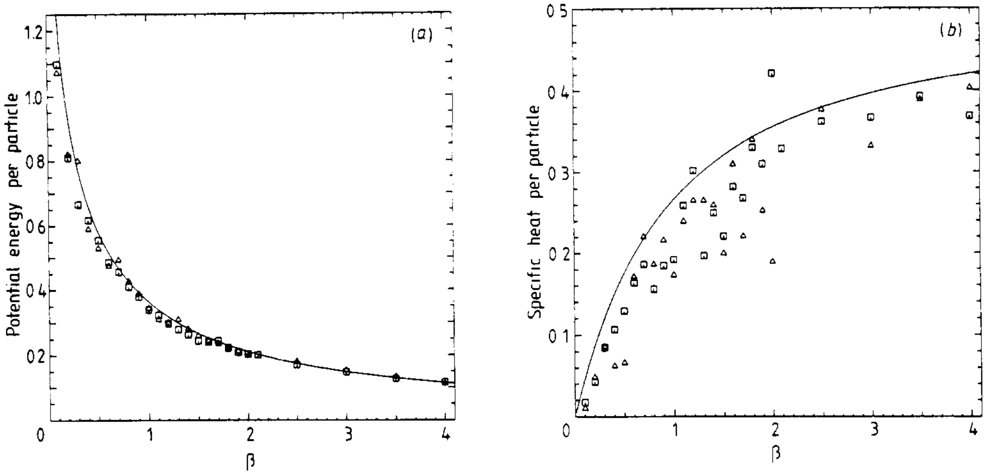

Figure 2. (*a*) Dependence of the mean potential energy per particle $U(\beta)$ for a system of $N = 99$ particles with inverse temperature β for Noise = 0.1 (open triangles) and Noise = 0.2 (open squares) compared with the theoretical prediction (3.5) (full curve). (*b*) The same for the mean specific heat per particle $C(\beta)$.

spacing distribution, the Δ statistics and the number variance and use the approximate formulae (2.7), (2.12) and (2.9), respectively. As a measure of the differences between the experimental histograms for the spacing distribution and the theoretical prediction we calculate the χ^2 values, giving the sum of the relative variances between theory and experiment over all bins of the experimental histogram containing ten and more spacings. The remaining sparse bins we accumulated. As we sample the particle spacings for 100 uncorrelated configurations, the total number of spacings is 9900. With this statistics we do not yet expect to see differences between Wigner's surmise (2.7) and the exact spacing distribution. We also used the conjectured distribution for $P_\beta(S)$ (4.1) and searched for the $\beta = \beta_{\text{histo}}$ that minimises the χ^2 values for the experimental histograms. Figure 3 shows the experimental histograms for $P(S)$ in all three cases. It is compared with the Wigner surmise (2.7) and our conjecture (4.1). The deviation from experiment comes out slightly smaller for the latter one, but both theoretical predictions follow the experiment quite closely. Taking β in the asymptotically correct distribution (4.1) as a fitting parameter, calling it β_{histo} and comparing it with the inverse temperature β of the Coulomb gas, we see a good correspondence for $\beta = 2$ but slight deviations for $\beta = 1$ and 4. This feature will be seen more clearly in the next section.

Turning now to the Δ statistics, we compare the experimental results sampled over 100 sets of particle positions with the theoretical prediction (2.12). We find good agreement for $\beta = 1$, but excellent agreement for $\beta = 2$ and 4 even down to $L \simeq 5$ although the theoretical predictions do not take into account terms of $O(1/L)$ (see figure 4). This is in complete agreement with the results for the time-reversal-invariant kicked top ($\beta = 1$, 4) with chaotic classical limit (Scharf *et al* 1988).

Finally we turn to the number variance $\Sigma_\beta^2(L)$. The fluctuations of the experimental values around the theoretical values (2.9) were larger than the fluctuations for the Δ statistics. Nevertheless, the three cases were clearly distinguishable from each other and results were not in contradiction with theoretical predictions.

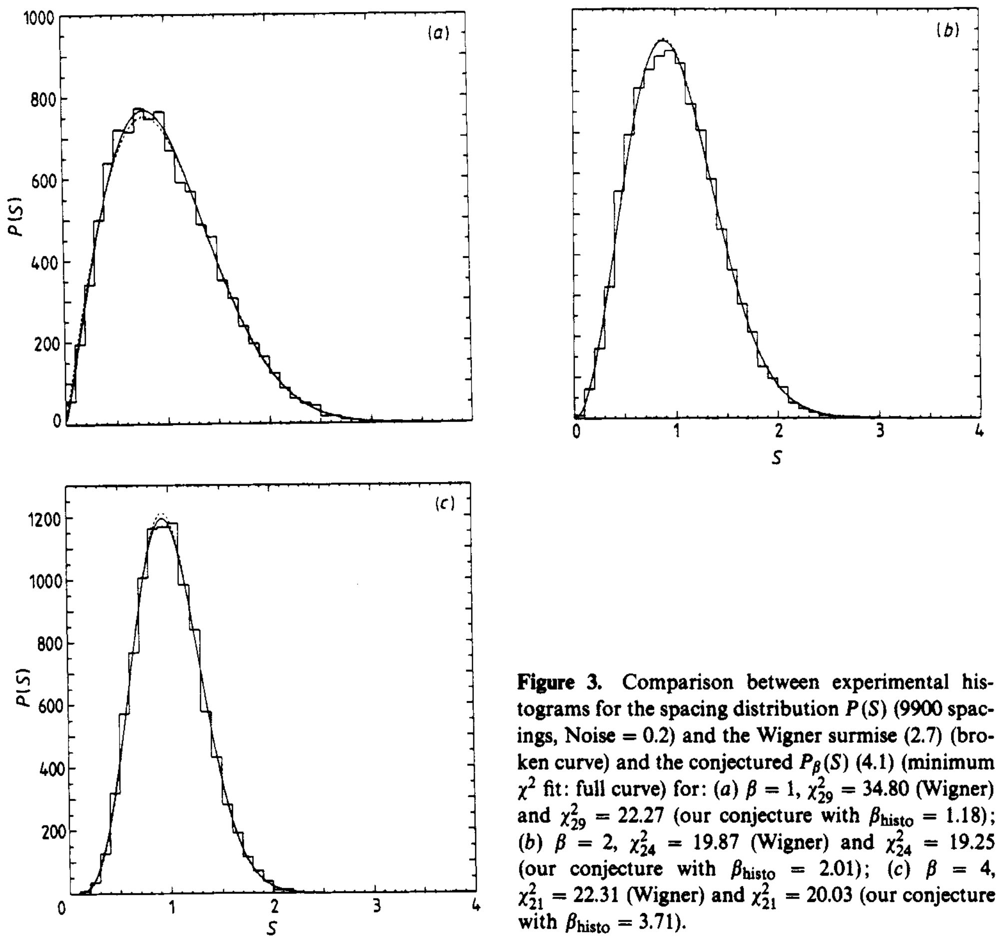

Figure 3. Comparison between experimental histograms for the spacing distribution $P(S)$ (9900 spacings, Noise = 0.2) and the Wigner surmise (2.7) (broken curve) and the conjectured $P_\beta(S)$ (4.1) (minimum χ^2 fit: full curve) for: (*a*) $\beta = 1$, $\chi^2_{29} = 34.80$ (Wigner) and $\chi^2_{29} = 22.27$ (our conjecture with $\beta_{\text{histo}} = 1.18$); (*b*) $\beta = 2$, $\chi^2_{24} = 19.87$ (Wigner) and $\chi^2_{24} = 19.25$ (our conjecture with $\beta_{\text{histo}} = 2.01$); (*c*) $\beta = 4$, $\chi^2_{21} = 22.31$ (Wigner) and $\chi^2_{21} = 20.03$ (our conjecture with $\beta_{\text{histo}} = 3.71$).

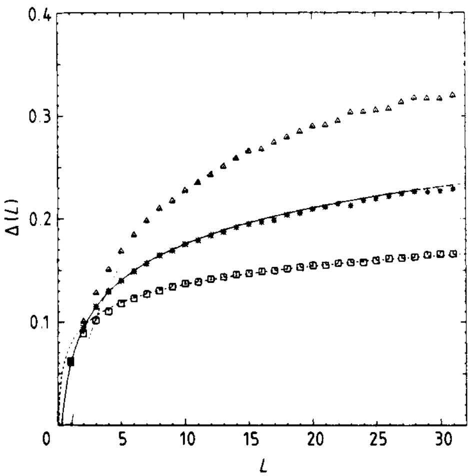

Figure 4. Comparison between experimental Δ statistics (9900 values, Noise = 0.2) and the theoretical prediction (2.12) for $\beta = 1$ (open triangles and dotted curve), $\beta = 2$ (asterisks and full curve) and $\beta = 4$ (open squares and broken curve).

7. Results for intermediate β

We come to the main part of our investigation, namely the behaviour of the interesting statistical properties of the Coulomb gas with inverse temperature β in the whole interval $(0,4]$. The results for the potential energy $U(\beta)$ and the specific heat $C(\beta)$ in section 5 have shown that the Monte Carlo simulation is reliable for the whole interval with the possible exception of $\beta \leq 0.2$.

First we compare histograms of experimental spacing distribution for intermediate inverse temperature β of the Coulomb gas with the conjectured spacing distribution (4.1). By minimising the χ^2 value we determine an experimental β value called β_{histo}. In addition, we find error bars for β_{histo} by fitting for χ^2 values that correspond to 1% confidence (Abramowitz and Stegun 1965). We now have the possibility of comparing β_{histo} with the *true* β, which is, of course, not known for intermediate spacing distributions stemming from quantum spectra. In figure 5 we present a few examples of histograms together with their minimum χ^2 fitting distribution and the 1% confidence fittings. These examples illustrate the fact that it is possible to fit experimental spacing distributions over the whole range of physically interesting β values with sufficiently small deviations when using the proposed distribution (4.1). Only for $0 < \beta \leq 0.2$ does the conjectured distribution not seem to work well, but even here its deviations from the experimental histograms are not worse than those for the Brody distribution.

Figure 6 collects these results. It shows the dependence of β_{histo} upon the inverse temperature $\beta = \beta_{\text{therm}}$ over the range $0 < \beta \leq 4$. As a good approximation, $\beta_{\text{histo}} \approx \beta_{\text{therm}}$ holds and can be used for practical purposes if the exact $\beta = \beta_{\text{therm}}$ is unknown, as is usually the case. Two deviations from this simple relation can be noticed upon closer inspection. One is a slight underestimation of the true β when using β_{histo} for large values of β. The other is a slight deviation for small values of β which amounts to an overestimation of the true β by using β_{histo}. Nevertheless figure 6 does show that the proposed spacing distribution (4.1) can be used over the whole range of physically relevant β values to get an approximation β_{histo} that describes the physics underlying the spacing distribution. For example, it can be used to find the scaling behaviour of the spacing distribution of systems with finite-dimensional Hilbert space that show a localisation–delocalisation transition for the eigenfunctions, and compare it with the scaling behaviour of the eigenfunctions themselves (Casati *et al* 1990).

Finally we check our conjecture (4.2) concerning the β dependence of the Δ statistics, which simply states

$$\Delta_\beta(L) \sim \frac{1}{\beta}\ln(L) + \text{constant}. \tag{7.1}$$

Therefore a value for β called β_{delta} can be determined by a least-squares fit of the experimental Δ statistics, which we restricted to the interval $5 \leq L \leq 31$. The dependence of β_{delta} on $\beta = \beta_{\text{therm}}$ is shown in figure 7. The error bars are, in this case, not as easy to find as in the case of β_{histo}, but they are quite large (± 0.2). Once again we have as a good approximation $\beta_{\text{delta}} \approx \beta_{\text{therm}}$. But the systematic deviations from this simple behaviour are the same as in the previous case: overestimation of the true β for small values and underestimation for large values of β.

Figure 8 shows the dependence of the constant term in (7.1) or (4.2), called $\Delta_\beta(1)$ for short, on β and compares it with the three theoretical constants given in (2.12). For small β this constant becomes more and more questionable, since the assumption

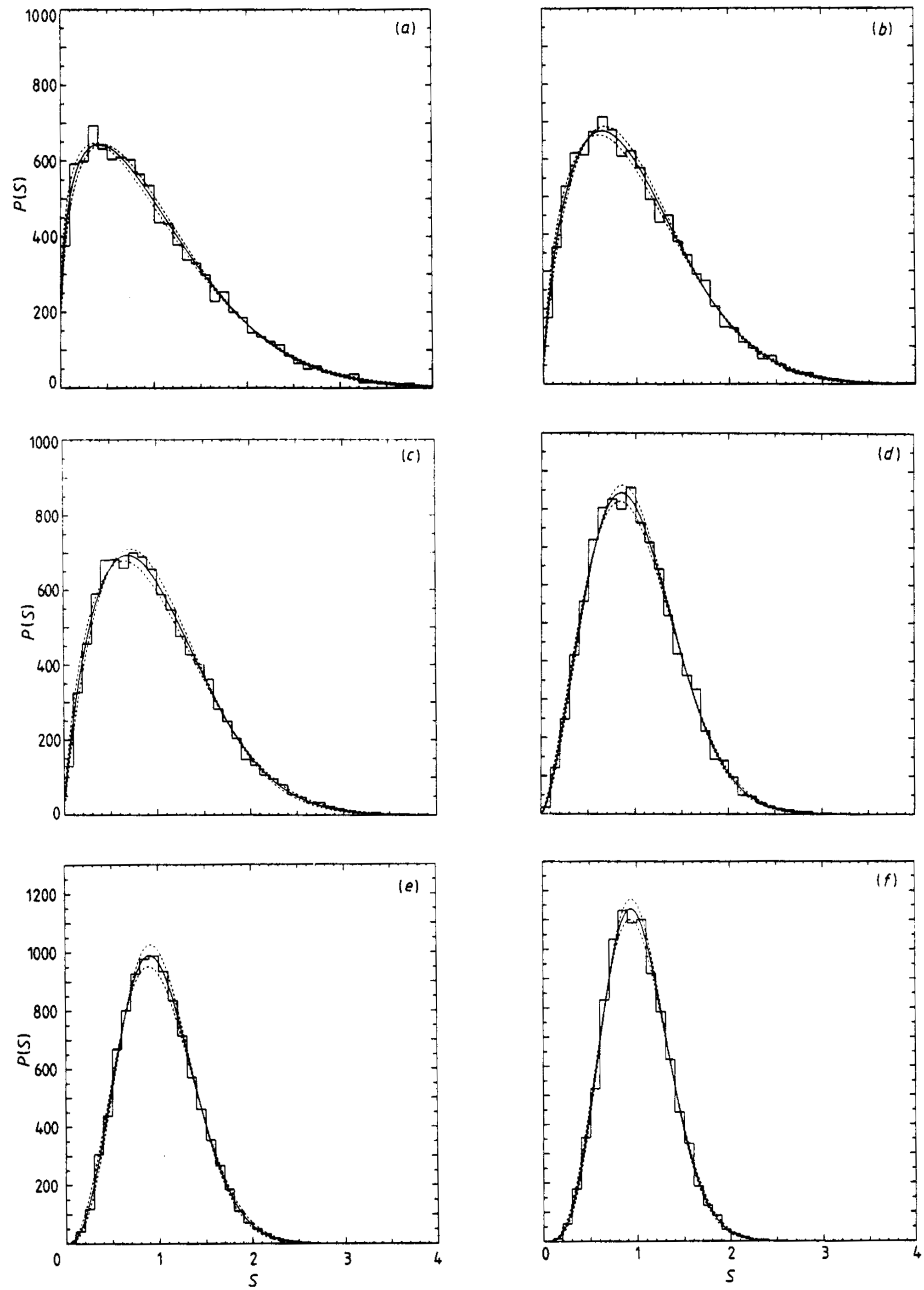

Figure 5. Comparison between experimental histograms for $P(S)$ for intermediate β (9900 spacings, Noise = 0.1) and the conjectured $P_\beta(S)$ (4.1) for minimum χ^2 fit (full curve) and 1%-confidence fits (broken curves) for: (*a*) $\beta = 0.2$, $\beta_{\text{histo}} = 0.35$ ($\chi^2_{38} = 44.49$); (*b*) $\beta = 0.5$, $\beta_{\text{histo}} = 0.68$ ($\chi^2_{33} = 34.35$); (*c*) $\beta = 0.7$, $\beta_{\text{histo}} = 0.79$ ($\chi^2_{32} = 24.99$); (*d*) $\beta = 1.5$, $\beta_{\text{histo}} = 1.57$ ($\chi^2_{28} = 31.44$); (*e*) $\beta = 2.5$, $\beta_{\text{histo}} = 2.39$ ($\chi^2_{24} = 16.31$); (*f*) $\beta = 3.5$, $\beta_{\text{histo}} = 3.31$ ($\chi^2_{21} = 18.51$).

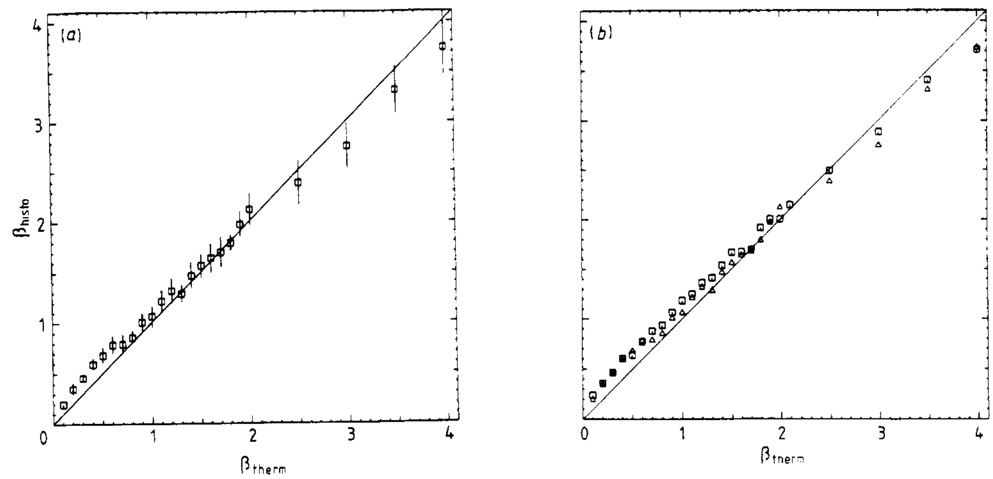

Figure 6. Dependence of β_{histo}, gained from minimum χ^2 fit of experimental histograms using conjectured spacing distribution (4.1), upon $\beta = \beta_{\text{therm}}$ for: (*a*) Noise=0.1 with error bars from 1%-confidence fits; (*b*) Noise=0.1 (open triangles) and Noise = 0.2 (open squares).

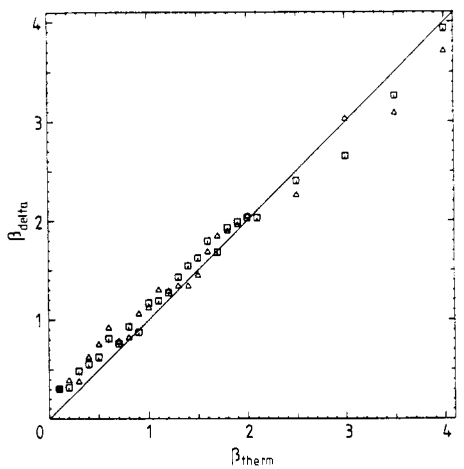

Figure 7. Dependence of β_{delta} gained from a least-square fit of experimental Δ statistics using conjecture (4.2) on $\beta = \beta_{\text{therm}}$ for Noise = 0.1 (open triangles) and Noise = 0.2 (open squares).

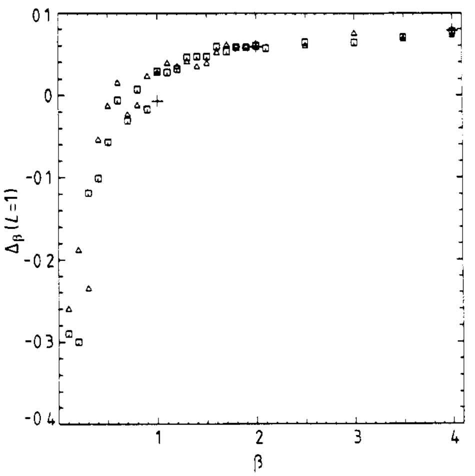

Figure 8. Dependence of Δ_β ($L = 1$) gained from a least-square fit of experimental Δ statistics using conjecture (4.2) on $\beta = \beta_{\text{therm}}$ for Noise = 0.1 (open triangles) and Noise = 0.2 (open squares). Large crosses indicate theoretical values from (2.12).

(7.1) is baseless for $\beta = 0$, as we already mentioned. But for $2 \le \beta \le 4$ the numerical results show an interpolating behaviour between the theoretical values at $\beta = 2$ and 4, with which they nearly coincide.

8. Summary and discussion

We have proposed a new approximation formula (4.1) for the spacing distribution of particle positions of Dyson's two-dimensional Coulomb gas on the unit circle with inverse temperature β, and have shown its usefulness with the help of Monte Carlo simulations. For the range $0.2 \leq \beta \leq 2.5$ the inverse temperature β_{histo}, found by least χ^2 fit of experimental spacing distributions with the conjectured distribution (4.1), showed satisfactory agreement with the 'true β'. Slight overestimation for small β and underestimation for large β can be taken into account to gauge the procedure in the range $0.1 \leq \beta \leq 4$, if larger accuracy in determining β is needed. Comparison with the Brody distribution shows that distribution (4.1) is superior not only for $\beta > 1$, where the Brody distribution has the wrong asymptotic behaviour for $S \to \infty$, but even for $0.1 < \beta \leq 1$.

We also showed that it is possible to find the inverse temperature β from the spectral stiffness (measured by the Δ statistics) of the particle positions of the gas. This was done with the help of conjecture (7.1), namely: $\Delta_\beta(L) \sim (1/\beta)\ln(L) + \text{constant}$. Both these ways of measuring β are successful in the range $0.1 < \beta \leq 4$, but fail in the high-temperature limit $\beta \to 0$.

The partition function (2.2) of the Coulomb gas coincides, for $\beta = 1, 2$ and 4, with the normalisation integrals of the eigenphase distributions for the three circular ensembles of random unitary matrices (COE, CUE and CSE, respectively). The usefulness of these three ensembles for describing spectral properties of kicked quantum systems with completely chaotic classical limit is well known (Izrailev 1986, Frahm and Mikeska 1986, Izrailev 1987, Kuś *et al* 1987, Scharf *et al* 1988, Haake *et al* 1988). Many examples of kicked dynamics have been found which fall into one of these three universality classes, depending on anti-unitary symmetries of the dynamics.

We suggest comparing properties of the Coulomb gas for intermediate β values, other than 1, 2 or 4, with spectral properties of kicked quantum systems with discrete spectra and chaotic classical limit, which nevertheless do not fall into one of the mentioned three universality classes. Fitting experimental spacing distributions with the conjectured distribution (4.1) and finding β_{histo} enables the determination of the 'true β' of the spectrum. This can be used, for example, to scrutinise changes in the spectral statistics upon changing parameters of the dynamics. If a quantum system shows, for example, dynamic localisation—like the kicked rotator on the torus for a spin-0 (Izrailev 1986, Feingold and Fishman 1987, Izrailev 1987, Frahm and Mikeska 1988) or for a spin-$\frac{1}{2}$ particle (Scharf 1989)—a reduction of β from the maximal possible value (1, 2, 4) is observed. The dependence of this reduction upon the degree of localisation can now be measured free of arbitrariness. Proposed scaling behaviour (Izrailev 1988) of spectral and localisation properties (Casati *et al* 1990) can be compared.

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Note added in proof. In the meantime Caurier and Grammatikos (1989) found quartic level repulsion ($\beta = 4$) also for an autonomous, fermion shell model Hamiltonian, therby completing the picture given at the end of section 2. Dietz and Haake (1990) calculated the Taylor expansion for the spacing distribution up to order S^{42} for $\beta = 1,4$ and order S^{32} for $\beta = 2$!

References

Abramowitz M and Stegun I (ed) 1965 *Handbook of Mathematical Functions* (New York: Dover)
Berry M V 1985 *Proc. R. Soc.* A **400** 229
Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
—— 1986 *J. Phys. A: Math. Gen.* **19** 649
Berry M V and Tabor M 1977 *Proc. R. Soc.* A **356** 375
Binder K and Heermann D W 1988 *Monte Carlo Simulation in Statistical Physics* (Berlin: Springer)
Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
Casati G, Guarneri I, Izrailev F M and Scharf R 1990 *Phys. Rev. Lett.* **64** 5
Caurier E and Grammaticos B 1989 *Phys. Lett.* **136A** 387
Dietz B 1989 private communication
Dietz B and Haake F 1990 *Z. Phys.* B, submitted
Dyson F J 1962a *J. Math. Phys.* **3** 140
—— 1962b *J. Math. Phys.* **3** 157
Eckhardt B 1988 *Phys. Rep.* **163** 205
Feingold M and Fishman S 1987 *Physica* **25D** 181
Frahm H and Mikeska M J 1986 *Z. Phys.* B **65** 249
—— 1988 *Phys. Rev. Lett.* **60** 3; **61** 378
Haller E, Köppel H and Cederbaum L S 1984 *Phys. Rev. Lett.* **52** 1665
Haake F, Kuś M and Scharf R 1987 *Z. Phys.* B **65** 381
Izrailev F M 1984 Preprint INP84-63 Novosibirsk
—— 1986 *Phys. Rev. Lett.* **56** 541
—— 1987 *Phys. Lett.* **125A** 250
—— 1988 *Phys. Lett.* **134A** 13
Kamien R D, Politzer H D and Wise M B 1988 *Phys. Rev. Lett.* **60** 1995
Kuś M, Mostowski J and Haake F 1988 *J. Phys. A: Math. Gen.* **21** L1073
Kuś M, Scharf R and Haake F 1987 *Z. Phys.* B **66** 129
Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic)
Nakamura K, Okazaky Y and Bishop A R 1986 *Phys. Rev. Lett.* **57** 5
Pandey A 1979 *Ann. Phys., NY* **119** 170
Robnik M 1987 *J. Phys. A: Math. Gen.* **20** L495
Scharf R, Dietz B, Kuś M, Haake F and Berry M V 1988 *Europhys. Lett.* **5** 383
Scharf R 1989 *J. Phys. A: Math. Gen.* **22** 4223
Seligman T H, Verbaarschot J J M and Zirnbauer M R 1984 *Phys. Rev. Lett.* **53** 215
Yukawa T 1986 *Phys. Lett.* **116A** 227